ACCESS TO SCIENCE, ENGINEERING AND AGRICULTURE: MATHEMATICS 1

MATH00030

TRIMESTER 1 2024/2025

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Contents

5. Trigonometry

5.1. **Parity and Co-Function Identities**.

In Section 4.6 of Chapter 4 we looked at how to calculate trigonometric functions of values that lie outside the range $0 \leqslant \theta \leqslant \frac{\pi}{2}$. There we used a geometric approach which involved examining a unit circle. In this chapter we will look at algebraic methods of calculating these. What this means is that instead of looking at pictures we will use formulae.

Some of the formulae we will present in this chapter are quite hard to prove and in this case we will concentrate on using them rather than proving them. On the other hand, for some of them, it is easy to see where they come from and in these cases we will indicate how they can be proved.

Let us start with the *parity* identities. The reason for the name is that in Mathematics the parity of a number tells you whether it is even or odd. We also use even and odd to describe functions. An odd function is one for which $f(-x) = -f(x)$

and an even function is one for which $f(-x) = f(x)$. The formulae in Table 1 tell us whether each of the trigonometric functions are even or odd.

$\sin(-\theta) = -\sin(\theta)$	$\cos(-\theta) = \cos(\theta)$
$\tan(-\theta) = -\tan(\theta)$	$\cot(-\theta) = -\cot(\theta)$
$\operatorname{cosec}(-\theta) = -\operatorname{cosec}(\theta)$	$\sec(-\theta) = \sec(\theta)$

TABLE 1. Parity identities.

Remark 5.1.1. The first two of these identities can be seen immediately from the graphs of $y = \sin(\theta)$ and $y = \cos(\theta)$. The remaining identities then follow from the definitions of tan, cot, cosec and sec.

Here are some examples to show how they can be used.

Example 5.1.2. Find $\sin\left(\frac{11\pi}{6}\right)$.

Here we will first use the fact that the sine function repeats every 2π. Thus $\sin\left(\frac{11\pi}{6}\right) = \sin\left(\frac{11\pi}{6} - 2\pi\right) = \sin\left(-\frac{\pi}{6}\right)$. We can now use our table of common values and $\sin(-\theta) = -\sin(\theta)$ to obtain $\sin\left(-\frac{\pi}{6}\right) = -\sin\left(\frac{\pi}{6}\right) = -\frac{1}{2}$. Hence $\sin\left(\frac{11\pi}{6}\right) = -\frac{1}{2}$.

Example 5.1.3. Find $\cos\left(\frac{7\pi}{4}\right)$.

Again we will first use the fact that the cosine function repeats every 2π. Thus $\cos\left(\frac{7\pi}{4}\right) = \cos\left(\frac{7\pi}{4} - 2\pi\right) = \cos\left(-\frac{\pi}{4}\right)$. We can now use our table of common values and $\cos(-\theta) = \cos(\theta)$ to obtain $\cos\left(-\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$. Hence $\cos\left(\frac{7\pi}{4}\right) = \frac{1}{\sqrt{2}}$.

Example 5.1.4. Find $\tan\left(\frac{2\pi}{3}\right)$.

Here we will first use the fact that the tangent function repeats every π. Thus $\tan\left(\frac{2\pi}{3}\right) = \tan\left(\frac{2\pi}{3} - \pi\right) = \tan\left(-\frac{\pi}{3}\right)$. We can now use our table of common values and $\tan(-\theta) = -\tan(\theta)$ to obtain $\tan\left(-\frac{\pi}{3}\right) = -\tan\left(\frac{\pi}{3}\right) = -\sqrt{3}$. Hence $\tan\left(\frac{2\pi}{3}\right) = -\sqrt{3}$.

The next identities we will state are the *co-function* identities. A function f is a co-function to another function g if $f(\theta) = g(\phi)$ whenever $\theta + \phi = \frac{\pi}{2}$. In fact this is where the 'co' comes from in cosine, cosecant and cotangent, since these functions are co-functions with the sine, secant and tangent functions, respectively.

$\sin\left(\frac{\pi}{2} - \theta\right) = \cos(\theta)$	$\cos\left(\frac{\pi}{2} - \theta\right) = \sin(\theta)$
$\tan\left(\frac{\pi}{2} - \theta\right) = \cot(\theta)$	$\cot\left(\frac{\pi}{2} - \theta\right) = \tan(\theta)$
$\operatorname{cosec}\left(\frac{\pi}{2} - \theta\right) = \sec(\theta)$	$\sec\left(\frac{\pi}{2} - \theta\right) = \operatorname{cosec}(\theta)$

TABLE 2. Co-function identities.

Remark 5.1.5. Perhaps the easiest way to see where these identities come from is to go back to a right angled triangle.

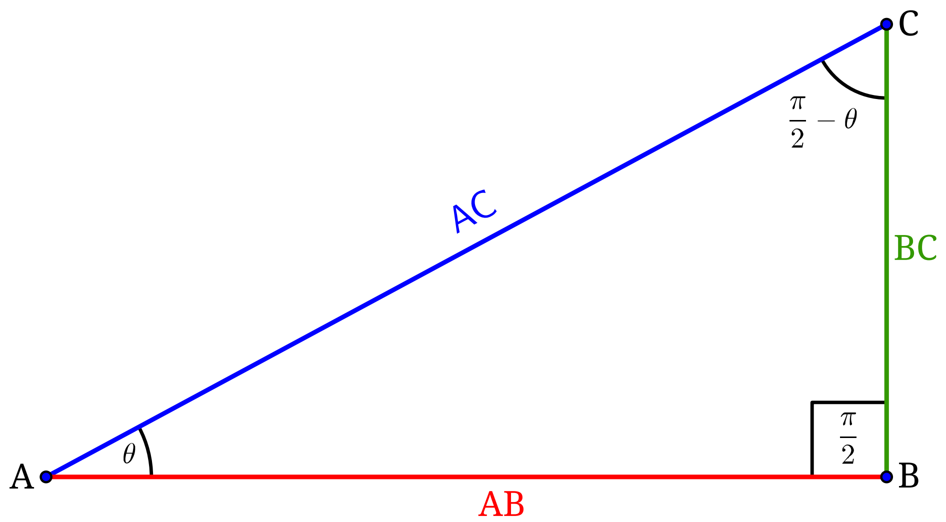

FIGURE 1. How to demonstrate that $\sin\left(\frac{\pi}{2} - \theta\right) = \cos(\theta)$.

For example, if we look at Figure 1, we have $\sin\left(\frac{\pi}{2} - \theta\right) = \frac{AB}{AC} = \cos(\theta)$.

The other co-function identities can be obtained in a similar manner.

Sometimes these can give an alternative method of calculating trig functions. Here are a couple of examples of this.

Example 5.1.6. Find $\cos\left(\frac{5\pi}{6}\right)$ (note this is the same question as in Example 4.6.7 in Chapter 4.

Here we will first use $\cos(\theta) = \sin\left(\frac{\pi}{2} - \theta\right)$.

We have $\cos\left(\frac{5\pi}{6}\right) = \sin\left(\frac{\pi}{2} - \frac{5\pi}{6}\right) = \sin\left(-\frac{\pi}{3}\right)$.

Next we will use $\sin(-\theta) = -\sin(\theta)$ and our table of common values to obtain $\sin\left(-\frac{\pi}{3}\right) = -\sin\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$. Hence $\cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}$, the same as we obtained in Example 4.6.7 in Chapter 4.

Example 5.1.7. Find $\tan\left(\frac{2\pi}{3}\right)$ (note this is the same question as in Example 5.1.4).

Here we will first use $\tan(\theta) = \cot\left(\frac{\pi}{2} - \theta\right)$.

We have $\tan\left(\frac{2\pi}{3}\right) = \cot\left(\frac{\pi}{2} - \frac{2\pi}{3}\right) = \cot\left(-\frac{\pi}{6}\right)$.

Next we will use $\cot(-\theta) = -\cot(\theta)$, the definition of cotangent and our table of common values to obtain $\cot\left(-\frac{\pi}{6}\right) = -\cot\left(\frac{\pi}{6}\right) = -\frac{1}{\tan\left(\frac{\pi}{6}\right)} = -\frac{1}{1/\sqrt{3}} = -\sqrt{3}$.

Hence $\tan\left(\frac{2\pi}{3}\right) = -\sqrt{3}$, the same as we obtained in Example 5.1.4.

5.2. Pythagorean Identities.

Next we come to the Pythagorean identities. These are particularly important and you should keep in mind the one involving sine and cosine if you are trying to solve any problem involving trigonometry.

$\sin^2(\theta) + \cos^2(\theta) = 1$
$\tan^2(\theta) + 1 = \sec^2(\theta)$
$1 + \cot^2(\theta) = \operatorname{cosec}^2(\theta)$

Table 3. Pythagorean identities.

Of course the last two identities only apply when the functions are defined (for example the second doesn't hold for $\theta = \frac{\pi}{2}$ since neither the tangent nor the secant are defined there).

As might be expected given their name, these identities are derived using Pythagoras' Theorem. Looking at Figure 1 again, we see that Pythagoras' Theorem says that $|BC|^2 + |AB|^2 = |AC|^2$. If we divide both sides of this equation by $|AC|^2$, we obtain $\frac{|BC|^2}{|AC|^2} + \frac{|AB|^2}{|AC|^2} = \frac{|AC|^2}{|AC|^2}$. Using Theorem 1.2.18 of Chapter 1 and cancelling the term on the right hand side, this may be rewritten $\left(\frac{|BC|}{|AC|}\right)^2 + \left(\frac{|AB|}{|AC|}\right)^2 = 1$. However from Figure 1 we see that $\sin(\theta) = \frac{|BC|}{|AC|}$ and $\cos(\theta) = \frac{|AB|}{|AC|}$. Thus $\sin^2(\theta) + \cos^2(\theta) = 1$. Of course we have only proved the identity for values of θ between 0 and $\frac{\pi}{2}$ but it does give you a good way of deriving it (while I will include it on a formula sheet in the exam, this may not be the case for all the exams you will take in the future).

I won't indicate how to prove the result for other values of θ but I will show how the other two results can be derived from this one (the arguments are valid whenever the respective identities are).

If we divide $\sin^2(\theta) + \cos^2(\theta) = 1$ by $\cos^2(\theta)$ we obtain $\frac{\sin^2(\theta)}{\cos^2(\theta)} + \frac{\cos^2(\theta)}{\cos^2(\theta)} = \frac{1}{\cos^2(\theta)}$. That is $\left(\frac{\sin(\theta)}{\cos(\theta)}\right)^2 + 1 = \left(\frac{1}{\cos(\theta)}\right)^2$. Using the definitions of the tangent and the secant this says that $\tan^2(\theta) + 1 = \sec^2(\theta)$, the second identity.

If we divide $\sin^2(\theta) + \cos^2(\theta) = 1$ by $\sin^2(\theta)$ we obtain $\frac{\sin^2(\theta)}{\sin^2(\theta)} + \frac{\cos^2(\theta)}{\sin^2(\theta)} = \frac{1}{\sin^2(\theta)}$. That is $1 + \left(\frac{\cos(\theta)}{\sin(\theta)}\right)^2 = \left(\frac{1}{\sin(\theta)}\right)^2$. Using the definitions of the cotangent and the cosecant this says that $1 + \cot^2(\theta) = \mathrm{cosec}^2(\theta)$, the third identity.

5.3. **Sine and Cosine Rules**.

In this section we will look at the sine and cosine rules. These rules are usually used when we have some information about a triangle (which will not in general be a right-angled triangle) and we want to find some more. For example we may know all the lengths of the sides of a triangle and we want to find the angles or we may know the lengths of two sides of a triangle and the included angle and we want to find the length of the remaining side and the sizes of the other two angles.

Although these rules are not difficult to prove, they are a little bit more complicated than the rules we have met up to now, so rather than prove them, we will state them and then go on to show how they can be used.

The sine and cosine rules are usually stated for a triangle where sizes of the angles are labelled A, B and C and the lengths of the sides opposite these angles are labelled a, b and c, as shown in Figure 2. Note that although I have drawn a triangle with an angle greater than $\frac{\pi}{2}$, this does not have to be the case, the triangle is completely arbitrary. The rules are as follows.

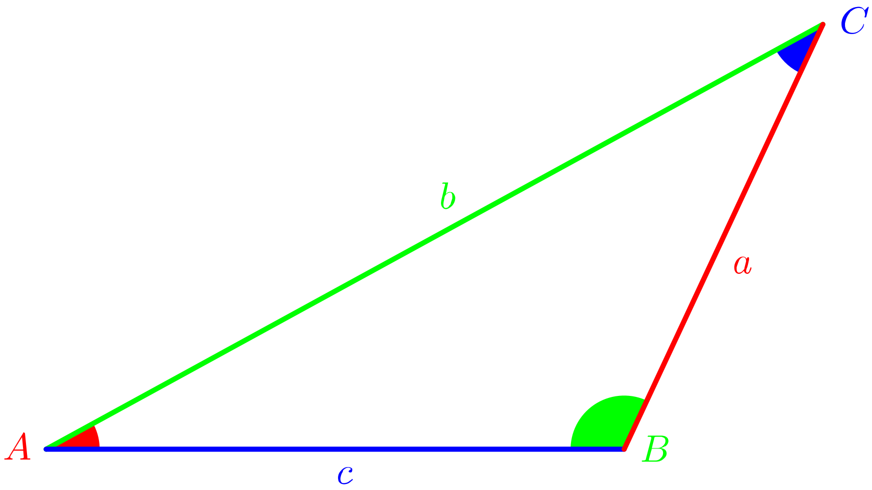

FIGURE 2. The triangle used in the sine and cosine rules.

Theorem 5.3.1 (The Sine Rule)**.** *Given a triangle as shown in Figure 2,*

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}.$$

Equivalently

$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c}.$$

Remark 5.3.2. Note that to use the sine rule we need to know the values of an angle and the opposite side (for example A and a) and the value of at least one more angle or side.

Theorem 5.3.3 (The Cosine Rule)**.** *Given a triangle as shown in Figure 2,*

$$a^2 = b^2 + c^2 - 2bc\cos(A).$$

Equivalently

$$b^2 = a^2 + c^2 - 2ac\cos(B)$$

or

$$c^2 = a^2 + b^2 - 2ab\cos(C).$$

Remark 5.3.4. Note that to use the cosine rule we either have to know the values of two sides and the included angle (for example A, b and c) or three sides.

Now let us have a look at some examples to see how these rules are used.

Example 5.3.5. Find all the angles in a triangle that has sides 5, 6 and 7.
Although it is not really necessary in this question, I think it is a good idea to get into the habit of drawing a quick diagram when doing these sorts of problems. I have shown this in Figure 3, where I have let $a = 7$, $b = 6$ and $c = 5$ (note it doesn't matter which sides you call a, b and c).

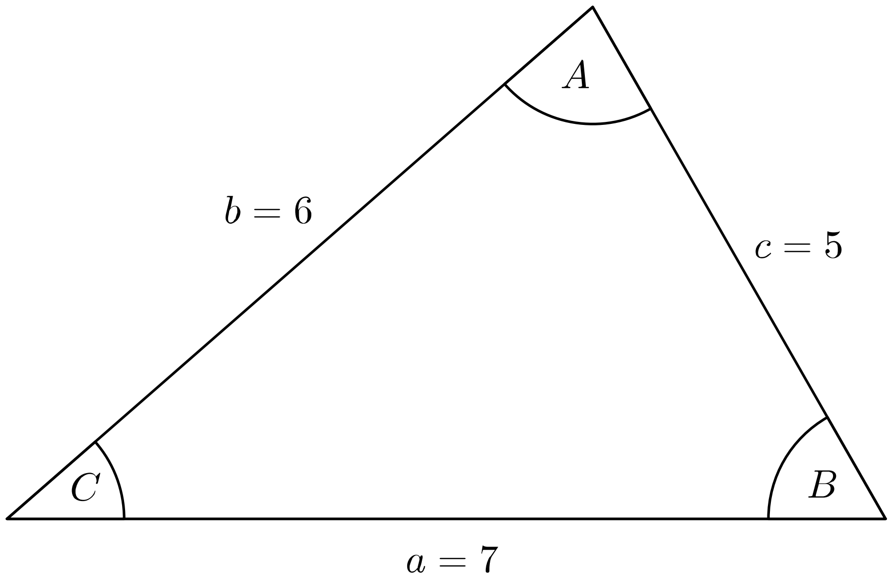

FIGURE 3. The triangle in Example 5.3.5.

It is not necessary to get the dimensions exact but they should be in the right ballpark, so you can see if your answers are 'reasonable'.

Now the question is should we use the sine rule or the cosine rule. The sine rule is only of use if we know an angle and an opposite side and at least another angle or side, so in this case we have to use the cosine rule. If we solve $a^2 = b^2+c^2-2bc\cos(A)$ for $\cos(A)$ we obtain $\cos(A) = \dfrac{b^2+c^2-a^2}{2bc}$ and on substituting for a, b and c, $\cos(A) = \dfrac{6^2+5^2-7^2}{2(6)(5)} = \dfrac{1}{5}$. Hence $A \simeq 78.46°$ (we are not doing calculus so it is fine to use degrees). Note this is only an approximation, so we use $\simeq$ rather than $=$.

Next on solving $b^2 = a^2+c^2-2ac\cos(B)$ for $\cos(B)$ we obtain $\cos(B) = \dfrac{a^2+c^2-b^2}{2ac}$ and on substituting for a, b and c, $\cos(B) = \dfrac{7^2+5^2-6^2}{2(7)(5)} = \dfrac{19}{35}$. Hence $B \simeq 57.12°$. Once we had found A, we could also have used the sine rule to find B. However there is a major drawback with the sine rule. The graph of the sine function is symmetric about the line $y = 90°$ (see Figure 24 in Chapter 4) and this means that $\sin(\theta) = \sin(180° - \theta)$. So, say we had found $\sin(B)$ using the sine rule, there would still be two possibilities for B, one greater than 90° and one less than 90° (unless of course $B = 90°$). So we would then have to decide which one of these Bs to choose. We could do this in several ways; perhaps we could argue that if it was bigger than 90° then the angles in the triangle would add up to more than 180° for example, so the correct B would be the one less than 90°. However some sort of argument is always required and my advice is that if you have a choice of which identity to use, you should use the cosine rule.

We could use the cosine rule again to find the last angle (or indeed the sine rule) but in this case it is easier to use the fact that the angles in a triangle sum to 180°. Thus $C = 180° - A - B \simeq 180° - 78.46° - 57.12° = 44.42°$. Note that in questions like this you should really use full calculator accuracy for A and B for otherwise you could introduce rounding errors.

Summing up, we have $A \simeq 78.46°$, $B \simeq 57.12°$ and $C = 44.42°$, where all the angles are correct to 2 d.p.

Example 5.3.6. Find the angle A in Figure 4.

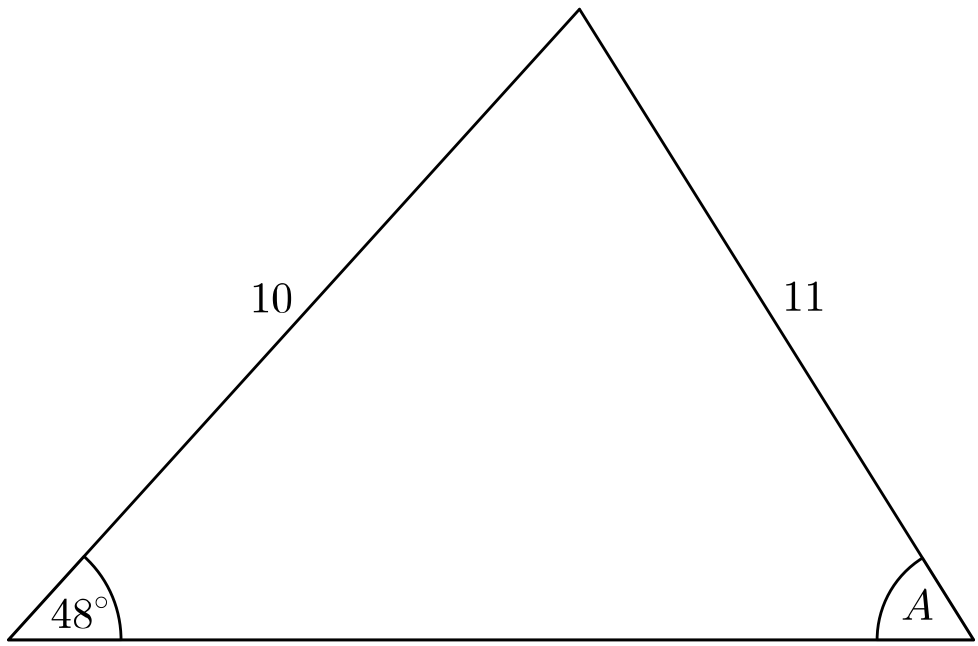

FIGURE 4. The triangle in Example 5.3.6.

In this case we neither have two sides and the included angle nor three sides, so we can't use the cosine rule and will have to use the sine rule. Using the sine rule we obtain $\dfrac{\sin(A)}{10} = \dfrac{\sin(48°)}{11}$. Thus $\sin(A) = \dfrac{10\sin(48°)}{11} \simeq 0.6756$. Hence $A \simeq 42.50°$ or $A \simeq 180° - 42.50° = 137.50°$. We now have to decide which of these values is correct. Of course if we look at Figure 4, it looks as if 42.50° is the correct value. However we want to prove this mathematically. Suppose that $A \simeq 137.50°$. Then the two angles we know add up to approximately $48° + 137.50° = 185.50°$ and this is too many degrees for a triangle. Hence we now know that $A \simeq 42.50°$, correct to 2 d.p.

Example 5.3.7. Find the length of the side a in Figure 5.

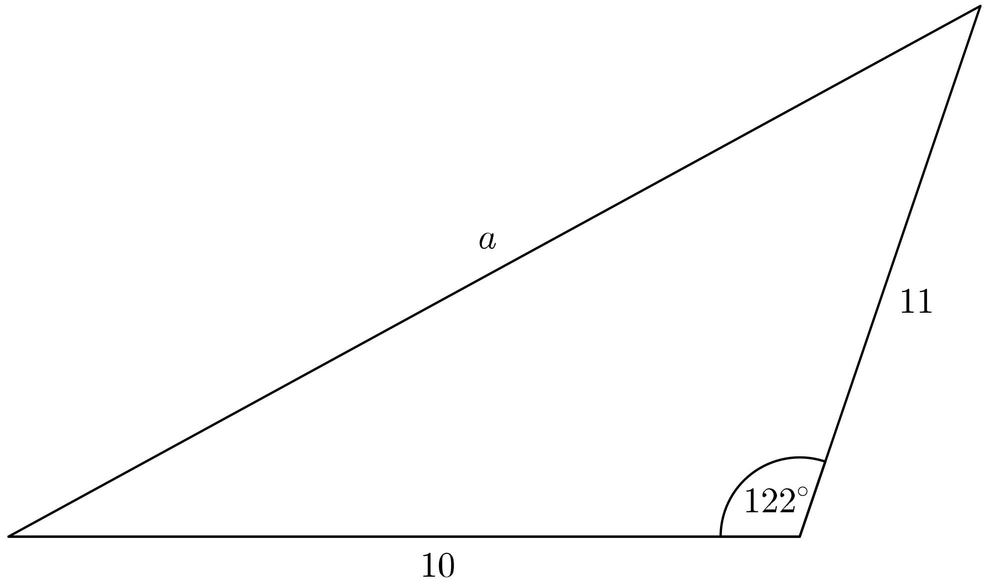

FIGURE 5. The triangle in Example 5.3.7.

Here we know two sides and an included angle so we are able to use the cosine rule. I will let $A = 122°$, $b = 11$ and $c = 10$. The angle has to be called A since it is opposite the side with length a but it does not matter if we let $b = 10$ and $c = 11$. In this case we can use the cosine in the form $a^2 = b^2 + c^2 - 2bc\cos(A)$ to obtain $a^2 = 11^2 + 10^2 - 2(11)(10)\cos(122°) \simeq 337.58$. Hence $a \simeq \sqrt{337.58} \simeq 18.37$ correct to 2 d.p. Note that I used the value of a^2 to full calculator accuracy to find a (although in this particular case it didn't affect the final answer to 2 d.p.).

5.4. **Sum and Difference Formulae**.

In this section we will look at formulae that allow us to express trigonometric functions of sums and differences of angles in terms of products, sums and differences of the trigonometric functions.

As was the case with the sine and cosine rules, I will not prove the results. Instead I will state them in Table 4 and then go on to show how they can be used.

Remark 5.4.1. These formulae can be more concisely stated as shown in in Table 5 and it is in this form that they will appear in the formula sheet in the exam.

$\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)$
$\sin(A-B) = \sin(A)\cos(B) - \cos(A)\sin(B)$
$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$
$\cos(A-B) = \cos(A)\cos(B) + \sin(A)\sin(B)$
$\tan(A+B) = \dfrac{\tan(A)+\tan(B)}{1-\tan(A)\tan(B)}$
$\tan(A-B) = \dfrac{\tan(A)-\tan(B)}{1+\tan(A)\tan(B)}$

TABLE 4. Sum and difference formulae.

$\sin(A\pm B) = \sin(A)\cos(B) \pm \cos(A)\sin(B)$
$\cos(A\pm B) = \cos(A)\cos(B) \mp \sin(A)\sin(B)$
$\tan(A\pm B) = \dfrac{\tan(A)\pm\tan(B)}{1\mp\tan(A)\tan(B)}$

TABLE 5. Concise form of sum and difference formulae.

Example 5.4.2. Find $\cos\left(\frac{\pi}{12}\right)$.
Here we will use $\cos(A-B) = \cos(A)\cos(B)+\sin(A)\sin(B)$ with $A = \frac{\pi}{4}$ and $B = \frac{\pi}{6}$.

$$\begin{aligned}\cos\left(\frac{\pi}{12}\right) &= \cos\left(\frac{\pi}{4}-\frac{\pi}{6}\right)\\ &= \cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{6}\right)+\sin\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{6}\right)\\ &= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2}\right)+\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right)\\ &= \frac{\sqrt{3}+1}{2\sqrt{2}}.\end{aligned}$$

Example 5.4.3. Find $\tan\left(\frac{5\pi}{12}\right)$.

Here we will use $\tan(A+B) = \frac{\tan(A)+\tan(B)}{1-\tan(A)\tan(B)}$ with $A = \frac{\pi}{4}$ and $B = \frac{\pi}{6}$.

$$\tan\left(\frac{5\pi}{12}\right) = \frac{\tan\left(\frac{\pi}{4}\right)+\tan\left(\frac{\pi}{6}\right)}{1-\tan\left(\frac{\pi}{4}\right)\tan\left(\frac{\pi}{6}\right)} = \frac{1+\frac{1}{\sqrt{3}}}{1-(1)\left(\frac{1}{\sqrt{3}}\right)} = \frac{\sqrt{3}+1}{\sqrt{3}-1}.$$

Remark 5.4.4. We can also simplify this as follows:

$$\frac{\sqrt{3}+1}{\sqrt{3}-1}\cdot\frac{\sqrt{3}+1}{\sqrt{3}+1} = \frac{3+2\sqrt{3}+1}{3-1} = \frac{4+2\sqrt{3}}{2} = 2+\sqrt{3}.$$

I will give full marks for $\tan\left(\frac{5\pi}{12}\right) = \frac{\sqrt{3}+1}{\sqrt{3}-1}$ however.

If we let $A = B = \theta$ in the first, third and fifth formulae in Table 4, we obtain the double angle formulae shown in Table 6 (the alternative expressions for $\cos(2\theta)$ are obtained using $\sin^2(\theta)+\cos^2(\theta) = 1$).

$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$
$\cos(2\theta) = \cos^2(\theta)-\sin^2(\theta)$ $= 2\cos^2(\theta)-1$ $= 1-2\sin^2(\theta)$
$\tan(2\theta) = \frac{2\tan(\theta)}{1-\tan^2(\theta)}$

TABLE 6. Double angle formulae.

Sometimes these give an easier solution compared to the sum and difference formulae.

Example 5.4.5. Find $\sin\left(\frac{2\pi}{3}\right)$.

Here we can let $\theta = \frac{\pi}{3}$ in $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$ to obtain

$$\sin\left(\frac{2\pi}{3}\right) = 2\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{3}\right) = 2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right) = \frac{\sqrt{3}}{2}.$$

5.5. Half Angle Formulae .

In this section we will look at further trigonometric formulae which can be used to calculate values of trigonometric functions but we will also use them in the second semester when they will help us to integrate squares of trigonometric functions. The formulae are shown in Table 7.

$\sin^2(\theta) = \dfrac{1-\cos(2\theta)}{2}$
$\cos^2(\theta) = \dfrac{1+\cos(2\theta)}{2}$
$\tan^2(\theta) = \dfrac{1-\cos(2\theta)}{1+\cos(2\theta)}$

Table 7. Half angle formulae.

Remark 5.5.1. Note that these formulae can be derived from the two formulae $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$ in Table 4 and $\sin^2(\theta) + \cos^2(\theta) = 1$ in Table 3, together with the definition of the tangent function.

Here are a couple of examples showing how they can be used to calculate the values of trigonometric functions.

Example 5.5.2. Find $\sin\left(\frac{\pi}{8}\right)$.

Using $\sin^2(\theta) = \dfrac{1-\cos(2\theta)}{2}$ with $\theta = \dfrac{\pi}{8}$, we have

$$\sin^2\left(\frac{\pi}{8}\right) = \frac{1-\cos\left(2\times\frac{\pi}{8}\right)}{2} = \frac{1-\cos\left(\frac{\pi}{4}\right)}{2} = \frac{1-\frac{1}{\sqrt{2}}}{2} = \frac{\sqrt{2}-1}{2\sqrt{2}}.$$

Hence $\sin\left(\frac{\pi}{8}\right) = \sqrt{\dfrac{\sqrt{2}-1}{2\sqrt{2}}}$.

Example 5.5.3. Find $\tan\left(\frac{\pi}{12}\right)$.

Using $\tan^2(\theta) = \dfrac{1-\cos(2\theta)}{1+\cos(2\theta)}$ with $\theta = \dfrac{\pi}{12}$, we have

$$\tan^2\left(\frac{\pi}{12}\right) = \frac{1-\cos\left(2\times\frac{\pi}{12}\right)}{1+\cos\left(2\times\frac{\pi}{12}\right)} = \frac{1-\cos\left(\frac{\pi}{6}\right)}{1+\cos\left(\frac{\pi}{6}\right)} = \frac{1-\frac{\sqrt{3}}{2}}{1+\frac{\sqrt{3}}{2}}.$$

Hence $\tan\left(\frac{\pi}{12}\right) = \sqrt{\dfrac{1-\frac{\sqrt{3}}{2}}{1+\frac{\sqrt{3}}{2}}}$.

Remark 5.5.4.
• This expression can be simplified to $\tan\left(\frac{\pi}{12}\right) = \sqrt{7-4\sqrt{3}}$ by multiplying top and bottom of $\dfrac{1-\frac{\sqrt{3}}{2}}{1+\frac{\sqrt{3}}{2}}$ by $1-\frac{\sqrt{3}}{2}$. We also have $\sqrt{7-4\sqrt{3}} = 2-\sqrt{3}$ but while it is easy to see that $(2-\sqrt{3})^2 = 7-4\sqrt{3}$, the other direction is not so obvious!
• Often in maths there is more than one way to do a problem. In this case we could also use $\tan(A-B) = \dfrac{\tan(A)-\tan(B)}{1+\tan(A)\tan(B)}$ with $A = \dfrac{\pi}{4}$ and $B = \dfrac{\pi}{6}$.

$$\begin{aligned}
\tan\left(\frac{\pi}{12}\right) &= \tan\left(\frac{\pi}{4}-\frac{\pi}{6}\right) \\
&= \frac{\tan\left(\frac{\pi}{4}\right)-\tan\left(\frac{\pi}{6}\right)}{1+\tan\left(\frac{\pi}{4}\right)\tan\left(\frac{\pi}{6}\right)} \\
&= \frac{1-\frac{1}{\sqrt{3}}}{1+(1)\left(\frac{1}{\sqrt{3}}\right)} \\
&= \frac{\sqrt{3}-1}{\sqrt{3}+1} \\
&= \frac{\sqrt{3}-1}{\sqrt{3}+1}\cdot\frac{\sqrt{3}-1}{\sqrt{3}-1} \\
&= \frac{3-2\sqrt{3}+1}{3-1} \\
&= \frac{4-2\sqrt{3}}{2} \\
&= 2-\sqrt{3}
\end{aligned}$$

5.6. Sum and Product Identities.

I won't examine you on them in this course but I will include the sum to product and product to sum formulae here since they might come in useful in future courses.

The first set of formulae, which are shown in Table 8, express sums of trigonometric formulae in terms of products.
The second set of formulae, which are shown in Table 9, express products of trigonometric formulae in terms of sums.

$\sin(A)+\sin(B) = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$
$\sin(A)-\sin(B) = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$
$\cos(A)+\cos(B) = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$
$\cos(A)-\cos(B) = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$

TABLE 8. Sum to product formulae.

$\sin(A)\sin(B) = \frac{1}{2}\left[\cos(A-B)-\cos(A+B)\right]$
$\cos(A)\cos(B) = \frac{1}{2}\left[\cos(A-B)+\cos(A+B)\right]$
$\sin(A)\cos(B) = \frac{1}{2}\left[\sin(A+B)+\sin(A-B)\right]$
$\cos(A)\sin(B) = \frac{1}{2}\left[\sin(A+B)-\sin(A-B)\right]$

TABLE 9. Product to sum formulae.